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NONLINEAR VIBRATIONS OF A VISCOELASTIC PLATE WITH CONCENTRATED MASSES

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The problem of vibrations of a viscoelastic plate with concentrated masses is studied in a geometrically nonlinear formulation. In the equation of motion of the plate, the action of the concentrated masses is taken into account using Dirac δ-functions. The problem is reduced to solving a system of Volterra type ordinary nonlinear integrodifferential equations using the Bubnov–Galerkin method. The resulting system with a singular Koltunov–Rzhanitsyn kernel is solved using a numerical method based on quadrature formulas. The effect of the viscoelastic properties of the plate material and the location and amount of concentrated masses on the vibration amplitude and frequency characteristics is studied. A comparison is made of numerical calculation results obtained using various theories.

Key words: *viscoelastic plate, concentrated mass, nonlinear vibrations, Bubnov–Galerkin method, relaxation kernel.*

In mechanical engineering and the building and aviation industries, one often encounters problems of vibrations of plates and shells of composite materials with longitudinal and transverse ribs, cover strips or units of devices. In theoretical studies of such problems, it is reasonable to treat these attached elements as added masses rigidly connected to systems and concentrated at points. The effect of concentrated masses on the system has an inertial nature [1]. The vibrations of elastic systems with concentrated masses have been studied in a number of papers [1–6], in which problems have been solved in a linear formulation [1] or some properties of structural materials [2–6] have been considered. At the same time, problems of nonlinear vibrations of elastic plates and shells with concentrated masses have been studied insufficiently: there are only some papers devoted to this issue (see, for example, [7]).

As is known, most composite materials possess distinct viscoelastic properties [8]. The practice of using new materials possessing viscoelastic properties and an analysis of their dynamic behavior show that their strength is greatly affected by heterogeneities of the type of attached masses.

Much less attention has been given to the behavior of inertially inhomogeneous viscoelastic systems [9]. The problems have been considered using the differential Voigt model or the Boltzmann–Volterra integral model, in which the relaxation kernels are exponential kernels which do not describe real processes in shells and plates at the initial time [10].

A feature of the problem considered is that use of Bubnov–Galerkin method reduces the problem in both linear and nonlinear formulations to solving irreducible systems of integrodifferential equations with singular kernels, which are difficult to study. Therefore, in many papers, additional conditions are introduced for the coordinate functions (see, for example, [3]).

The purpose of the present work is to study nonlinear vibrations of a viscoelastic plate with concentrated masses.

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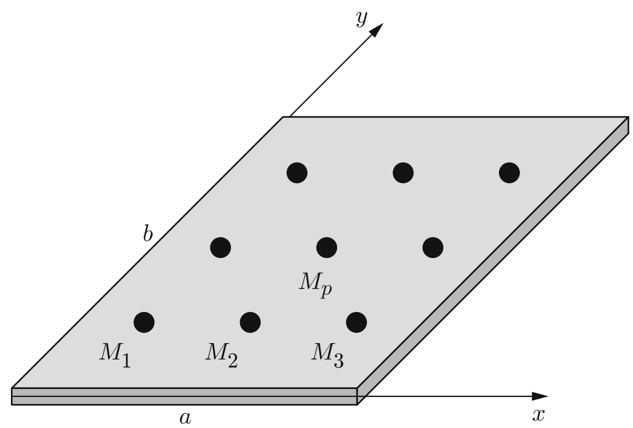

Fig. 1. Diagram of the plate with concentrated masses.

1. Mathematical Model. We consider a rectangular viscoelastic plate of thickness h with sides a and b, made of a homogeneous isotropic material and with concentrated masses M_p at the points (x_p, y_p), $p = 1, 2, \ldots, I$ (Fig. 1).

The relations between the stresses σ_x, σ_y, and τ_{xy} and the strains ε_x, ε_y, and γ_{xy} in the middle surface are written in the form [8, 10]

$$\sigma_x = \frac{E}{1-\mu^2}\,(1-R^*)(\varepsilon_x + \mu\varepsilon_y), \qquad \sigma_y = \frac{E}{1-\mu^2}\,(1-R^*)(\varepsilon_y + \mu\varepsilon_x), \qquad \tau_{xy} = \frac{E}{2(1+\mu)}\,(1-R^*)\gamma_{xy}, \tag{1}$$

where E is the elastic modulus, μ is Poisson's ratio, and R^* is the integral operator with the relaxation kernel $R(t)$:

$$R^*\varphi = \int_0^t R(t-\tau)\varphi(\tau)\,d\tau.$$

The relationship between the strains ε_x, ε_y, and γ_{xy} in the middle surface and the displacements u, v, and w in the x, y, and z directions is given by [11]

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\Big[\Big(\frac{\partial w}{\partial x}\Big)^2 - \Big(\frac{\partial w_0}{\partial x}\Big)^2\Big], \qquad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\Big[\Big(\frac{\partial w}{\partial y}\Big)^2 - \Big(\frac{\partial w_0}{\partial y}\Big)^2\Big],$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} - \frac{\partial w_0}{\partial x}\,\frac{\partial w_0}{\partial y}, \tag{2}$$

where $w_0 = w_0(x, y)$ is the initial deflection.

The effect of the concentrated masses on the viscoelastic plate has an inertial nature and is taken into account in the equation of motion by means of the Dirac δ-function [1]:

$$m(x, y) = \rho h + \sum_{p=1}^{I} M_p \delta(x - x_p)\delta(y - y_p) \tag{3}$$

(ρ is the density of the plate material).

Substituting (1) and (2) [in view of (3)] into the equations [11]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0,$$

$$-\frac{D}{h}\,(1-R^*)\nabla^4(w - w_0) + \frac{\partial}{\partial x}\Big(\sigma_x\,\frac{\partial w}{\partial x} + \tau_{xy}\,\frac{\partial w}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(\tau_{xy}\,\frac{\partial w}{\partial x} + \sigma_y\,\frac{\partial w}{\partial y}\Big)$$

$$+\,\frac{q}{h} - \Big(\rho + \frac{1}{h}\sum_{p=1}^{I} M_p\delta(x - x_p)\delta(y - y_p)\Big)\frac{\partial^2 w}{\partial t^2} = 0,$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = -\frac{1}{2}\,[L(w,w) - L(w_0,w_0)]$$

and introducing the stress function F in the form [11]

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y},$$

we obtain the Kármań type equations

$$\frac{D}{h}\,(1-R^*)\nabla^4(w-w_0) + \Big(\rho + \frac{1}{h}\sum_{p=1}^{I} M_p\delta(x-x_p)\delta(y-y_p)\Big)\frac{\partial^2 w}{\partial t^2} = L(w,F) + \frac{q}{h},$$
$$\frac{1}{E}\,\nabla^4 F = -\frac{1}{2}\,(1-R^*)[L(w,w) - L(w_0,w_0)], \tag{4}$$

where $D = Eh^3/[12(1-\mu^2)]$ is the flexural rigidity of the plate and q is the external load;

$$L(w,w) = 2\Big[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \Big(\frac{\partial^2 w}{\partial x \partial y}\Big)^2\Big], \quad L(w,F) = \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - 2\,\frac{\partial^2 w}{\partial x\,\partial y}\,\frac{\partial^2 F}{\partial x\,\partial y}.$$

Thus, the mathematical model of the problem of vibrations of viscoelastic plates with concentrated masses in a geometrically nonlinear formulation is described by the system of partial integrodifferential equations (4) with the corresponding boundary and initial conditions.

Using the results of [12], it is possible to obtain mathematical models for the problem of nonlinear vibrations of viscoelastic plates with concentrated masses:

$$\frac{D}{h}\,(1-R^*)\nabla^4(w-w_0) + \Big(\rho + \frac{1}{h}\sum_{p=1}^{I} M_p\delta(x-x_p)\delta(y-y_p)\Big)\frac{\partial^2 w}{\partial t^2}$$
$$= \frac{6D}{h^3ab}\,\nabla^2 w(1-R^*)\Big\{\int_0^a\int_0^b \Big[\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2 - \Big(\frac{\partial w_0}{\partial x}\Big)^2 - \Big(\frac{\partial w_0}{\partial y}\Big)^2\Big]dx\,dy\Big\} + \frac{q}{h}. \tag{5}$$

2. Choice of a Relaxation Kernel. Studies have shown that the integral relations of the hereditary viscoelasticity theory are equivalent to linear differential relations with constant coefficients if the kernel is the sum of exponential functions. At the same time, processing of experimental data have shown that the kernels containing one or several exponential terms are unsuitable for describing the properties of real materials; therefore, it is necessary to use more complex dependences based on weakly singular functions. In this case, the following circumstances should be taken into account. Results of studies of the creep and relaxation of viscoelastic materials suggest that the relaxation processes have extremely high intensity in the initial stage of tests. In this case, the rates of the processes are so high that their direct measurement at the initial time is impossible. Therefore, the processes should be considered dynamic and their rates should be considered equal to infinity [10].

This phenomenon can be described using weakly singular functions which provide for finite strains and stresses, in contrast to strongly singular functions, which yield infinitely large values. Weakly singular functions provide an adequate description of relaxation rates if they contain a sufficient number of rheological parameters, for example, the kernels proposed by Rabotnov, Rzhanitsyn, Koltunov, et al. [10].

In the following calculations, we will use the simple but fairly general weakly singular Koltunov–Rzhanitsyn kernel with three rheological parameters (A, β, and α) [10]:

$$R(t) = A\,e^{-\beta t}\,t^{\alpha-1}, \qquad 0 < \alpha < 1. \tag{6}$$

3. Algorithm for Numerical Solution of Nonlinear Problems of Dynamics of Viscoelastic Systems with Concentrated Masses. The numerical method proposed in [13] is suitable for solving nonlinear integrodifferential of the form

$$\sum_{n=1}^{N} a_{kn}\ddot{w}_n + \omega_k^2 w_k = X_k\Big(t, w_1, \ldots, w_N, \int_0^t \varphi_k(t,\tau,w_1(\tau),\ldots,w_N(\tau))\,d\tau\Big),$$
$$w_k(0) = w_{0k}, \qquad \dot{w}_k(0) = \dot{w}_{0k}, \qquad k = 1,2,\ldots,N, \tag{7}$$

where $w_k = w_k(t)$ are unknown functions of time, X_k and φ_k are continuous functions in the range of the arguments, and a_{kn} and ω_k are specified constants.

Many nonlinear dynamic problems of viscoelasticity [14–16], in particular, problems of the vibrations and dynamic stability of viscoelastic structures with concentrated masses are reduced to systems (7).

System (7) can be written in matrix form

$$A\ddot{w} + \omega^2 w = X\Big(t, w, \int_0^t \varphi(t,\tau,w(\tau))\,d\tau\Big), \qquad w(0) = w_0, \qquad \dot{w}(0) = \dot{w}_0, \tag{8}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \\ \ldots \\ w_N \end{pmatrix}, \qquad \omega^2 = \begin{pmatrix} \omega_1^2 & 0 & \ldots & 0 \\ 0 & \omega_2^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \omega_N^2 \end{pmatrix},$$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \ldots \\ X_N \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \ldots \\ \varphi_N \end{pmatrix}.$$

Solving system (8) for w, for the unknowns at the points $t_i = ih$ ($i = 0, 1, 2, \ldots$; h is the interpolation step), we obtain the recursive formula

$$w_{i+1} = \dot{w}_0 t_{i+1} + w_0 + A^{-1} \sum_{j=0}^{i} A_j (t_{i+1} - t_j) \Big[-\omega^2 w_j + X\Big(t_j, w_j, \sum_{k=0}^{j} B_k \varphi(t_j, t_k, w_k)\Big)\Big], \tag{9}$$

where A^{-1} is the matrix which is the reciprocal of the matrix A and A_j and B_k ($j = 0, 1, \ldots, i$; $k = 0, 1, \ldots, j$) are the nodes of the interpolation formula.

By using the Bubnov–Galerkin method, the corresponding problem in a two-dimensional formulation is reduced to the nonlinear integrodifferential equations

$$\sum_{n=1}^{N} \sum_{m=1}^{M} a_{klnm} \ddot{w}_{nm} + \omega_{kl}^2 w_{kl} = X_{kl}\Big(t, w_{11}, \ldots, w_{NM}, \int_0^t \varphi_{kl}(t, \tau, w_{11}(\tau), \ldots, w_{NM}(\tau))\,d\tau\Big),$$

$$w_{kl}(0) = w_{0kl}, \qquad \dot{w}_{kl}(0) = \dot{w}_{0kl}, \qquad k = 1, 2, \ldots, N, \qquad l = 1, 2, \ldots, M.$$

Introducing the matrices A, ω^2, X, and φ in the problem considered, for the dependence $w = w(t)$, we obtain the matrix equation (8), whose solution is found from the recursive relation (9).

4. Test Example. Let us consider the system of nonlinear integrodifferential equations

$$\ddot{u} + \lambda_1^2 u = p_x + \lambda_2 \int_0^t R(t-\tau)u(\tau)\,d\tau + \lambda_3 \int_0^t R_1(t-\tau)v(\tau)\,d\tau + \lambda_4 \int_0^t R_2(t-\tau)w^2(\tau)\,d\tau,$$

$$\ddot{v} + \varphi_1^2 v = p_y + \varphi_2 \int_0^t R_1(t-\tau)v(\tau)\,d\tau + \varphi_3 \int_0^t R(t-\tau)u(\tau)\,d\tau + \varphi_4 \int_0^t R_2(t-\tau)w^2(\tau)\,d\tau,$$

$$\ddot{w} + \omega_1^2 w = q + \omega_2 \int_0^t R_3(t-\tau)w(\tau)\,d\tau + \omega_3 w \int_0^t R(t-\tau)u(\tau)\,d\tau \tag{10}$$

$$+ \omega_4 w \int_0^t R_1(t-\tau)v(\tau)\,d\tau + \omega_5 \int_0^t R_2(t-\tau)w^2(\tau)\,d\tau$$

with the initial conditions $u(0) = 1$, $\dot{u}(0) = -\beta_1$, $v(0) = 1$, $\dot{v}(0) = -\beta_2$, $w(0) = 1$, and $\dot{w}(0) = -\beta_3$, where

$$R(t) = A\,\mathrm{e}^{-\beta_1 t}\,t^{\alpha-1}, \qquad R_1(t) = A_1\,\mathrm{e}^{-\beta_2 t}\,t^{\alpha_1-1}, \qquad R_2(t) = A_2\,\mathrm{e}^{-2\beta_3 t}\,t^{\alpha_2-1},$$

$$R_3(t) = A_3\,\mathrm{e}^{-\beta_3 t}\,t^{\alpha_3-1}, \qquad 0<\alpha<1, \quad 0<\alpha_1<1, \quad 0<\alpha_2<1, \quad 0<\alpha_3<1,$$

$$p_x = \Big(\beta_1^2 + \lambda_1^2 - \frac{\lambda_2 A}{\alpha}\,t^{\alpha}\Big)\,\mathrm{e}^{-\beta_1 t} - \frac{\lambda_3 A_1}{\alpha_1}\,\mathrm{e}^{-\beta_2 t}\,t^{\alpha_1} - \frac{\lambda_4 A_2}{\alpha_2}\,\mathrm{e}^{-2\beta_3 t}\,t^{\alpha_2},$$

$$p_y = \Big(\beta_2^2 + \varphi_1^2 - \frac{\varphi_2 A_1}{\alpha_1} t^{\alpha_1}\Big)\,\mathrm{e}^{-\beta_2 t} - \frac{\varphi_3 A}{\alpha}\,\mathrm{e}^{-\beta_1 t}\,t^{\alpha} - \frac{\varphi_4 A_2}{\alpha_2}\,\mathrm{e}^{-2\beta_3 t}\,t^{\alpha_2},$$

$$q = \Big(\beta_3^2 + \omega_1^2 - \frac{\omega_2 A_3}{\alpha_3}\,t^{\alpha_3}\Big)\,\mathrm{e}^{-\beta_3 t} - \frac{\omega_3 A}{\alpha}\,\mathrm{e}^{-(\beta_1+\beta_3)t}\,t^{\alpha} - \frac{\omega_4 A_1}{\alpha_1}\,\mathrm{e}^{-(\beta_2+\beta_3)t}\,t^{\alpha_1} - \frac{\omega_5 A_2}{\alpha_2}\,\mathrm{e}^{-2\beta_3 t}\,t^{\alpha_2}.$$

System (10) has an exact solution $u = \mathrm{e}^{-\beta_1 t}$, $v = \mathrm{e}^{-\beta_2 t}$, $w = \mathrm{e}^{-\beta_3 t}$ that satisfies the initial conditions. Doubly integrating system (10) subject to the initial conditions, we find the approximate values $u_n = u(t_n)$, $v_n = v(t_n)$, and $w_n = w(t_n)$ at the nodes $t_n = (n-1)\Delta t$, $n = 1, 2, \ldots$ [similar to formula (9)] from the relations

$$\begin{aligned}
u_n &= 1 - \beta_1 t_n + \sum_{i=0}^{n-1} B_i(t_n - t_i)\Big[p_x(t_i) - \lambda_1^2 u_i \\
&+ \sum_{k=0}^{i}\Big(\frac{\lambda_2 A}{\alpha}\,C_k\,\mathrm{e}^{-\beta_1 t_k}\,u_{i-k} + \frac{\lambda_3 A_1}{\alpha_1}\,C_{1k}\,\mathrm{e}^{-\beta_2 t_k}\,v_{i-k} + \frac{\lambda_4 A_2}{\alpha_2}\,C_{2k}\,\mathrm{e}^{-2\beta_3 t_k}\,w_{i-k}^2\Big)\Big],\\
v_n &= 1 - \beta_2 t_n + \sum_{i=0}^{n-1} B_i(t_n - t_i)\Big[p_y(t_i) - \varphi_1^2 v_i \\
&+ \sum_{k=0}^{i}\Big(\frac{\varphi_2 A_1}{\alpha_1}\,C_{1k}\,\mathrm{e}^{-\beta_2 t_k}\,v_{i-k} + \frac{\varphi_3 A}{\alpha}\,C_k\,\mathrm{e}^{-\beta_1 t_k}\,u_{i-k} + \frac{\varphi_4 A_2}{\alpha_2}\,C_{2k}\,\mathrm{e}^{-2\beta_3 t_k}\,w_{i-k}^2\Big)\Big],\\
w_n &= 1 - \beta_3 t_n + \sum_{i=0}^{n-1} B_i(t_n - t_i)\Big[q(t_i) - \omega_1^2 w_i \\
&+ \sum_{k=0}^{i}\Big(\frac{\omega_2 A_3}{\alpha_3}\,C_{3k}\,\mathrm{e}^{-\beta_3 t_k}\,w_{i-k} + \frac{\omega_3 A}{\alpha}\,w_i C_k\,\mathrm{e}^{-\beta_1 t_k}\,u_{i-k} \\
&+ \frac{\omega_4 A_2}{\alpha_2}\,w_i C_{1k}\,\mathrm{e}^{-\beta_2 t_k}\,v_{i-k} + \frac{\omega_5 A_2}{\alpha_2}\,C_{2k}\,\mathrm{e}^{-2\beta_3 t_k}\,w_{i-k}^2\Big)\Big],
\end{aligned} \tag{11}$$

where B_i, C_k, C_{1k}, C_{2k}, and C_{3k} are the coefficients of the quadrature trapezoid formula, $B_0 = h/2$, $B_i = h$ $(i = 1, 2, \ldots, n-1)$, and

$$C_0 = h^{\alpha}/2, \qquad C_i = h^{\alpha}[i^{\alpha} - (i-1)^{\alpha}]/2, \qquad C_k = h^{\alpha}[(k+1)^{\alpha} - (k-1)^{\alpha}]/2,$$

$$C_{l0} = h^{\alpha_l}/2, \qquad C_{li} = h^{\alpha_l}[i^{\alpha_l} - (i-1)^{\alpha_l}]/2, \qquad C_{lk} = h^{\alpha_l}[(k+1)^{\alpha_l} - (k-1)^{\alpha_l}]/2,$$

$$k = 1, 2, \ldots, n-1, \qquad l = 1, 2, 3.$$

Table 1 gives the results of calculations using formulas (11) in the range $t = 0$–0.8 with a step $\Delta t = 0.001$. The following initial data were used: $\lambda = 1.1$, $\lambda_1 = 1.2$, $\lambda_2 = 1.3$, $\lambda_3 = 1.4$, $\lambda_4 = 1.5$, $\varphi = 1.2$, $\varphi_1 = 1.3$, $\varphi_2 = 1.4$, $\varphi_3 = 1.5$, $\varphi_4 = 1.6$, $\omega = 1.3$, $\omega_1 = 1.4$, $\omega_2 = 1.5$, $\omega_3 = 1.6$, $\omega_4 = 1.7$, $\omega_5 = 1.8$, $A = 0.01$, $A_1 = 0.02$, $A_2 = 0.03$, $A_3 = 0.04$, $\beta = 0.25$, $\beta_1 = 0.26$, $\beta_2 = 0.27$, $\beta_3 = 0.28$, $\alpha = 0.05$, $\alpha_1 = 0.06$, $\alpha_2 = 0.07$, and $\alpha_3 = 0.08$.

TABLE 1

t	Solution		Δ
	Exact	Approximate	
0	1.000000	1.000000	—
0.1	0.974335	0.974237	$9.8 \cdot 10^{-5}$
0.2	0.949329	0.948868	$4.6 \cdot 10^{-4}$
0.3	0.924964	0.923942	10^{-3}
0.4	0.901225	0.899560	$1.6 \cdot 10^{-3}$
0.5	0.878095	0.875867	$2.2 \cdot 10^{-3}$
0.6	0.855559	0.853050	$2.5 \cdot 10^{-3}$
0.7	0.833601	0.831336	$2.0 \cdot 10^{-3}$
0.8	0.812207	0.810994	$1.2 \cdot 10^{-3}$

From Table 1 it follows that the error Δ of the calculations performed using the method described here coincides with the error of the quadrature formulas used and has the same order of smallness for the interpolation step.

5. Calculation of Nonlinear Vibrations of the Viscoelastic Plate with Concentrated Masses. Bubnov–Galerkin Method. Let the plate be simply-supported on all sides. The plate deflections w and w_0 are approximated by the function

$$w(x,y,t)=\sum_{n=1}^{N}\sum_{m=1}^{M} w_{nm}(t)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}, \qquad w_0(x,y)=\sum_{n=1}^{N}\sum_{m=1}^{M} w_{0nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}. \tag{12}$$

Substituting (12) into the second equation (4) and equating the coefficients of the identical harmonics of the trigonometric functions on both sides of this equation, we find the force function

$$F(x,y,t)=E\sum_{i,j=1}^{N}\sum_{r,s=1}^{M}(1-R^*)(w_{ir}w_{js}-w_{0ir}w_{0js})\Big(C_{irjs}\cos\frac{(i+j)\pi x}{a}\cos\frac{(r+s)\pi y}{b}$$

$$+A_{irjs}\cos\frac{(i+j)\pi x}{a}\cos\frac{(r-s)\pi y}{b}+D_{irjs}\cos\frac{(i-j)\pi x}{a}\cos\frac{(r+s)\pi y}{b}+B_{irjs}\cos\frac{(i-j)\pi x}{a}\cos\frac{(r-s)\pi y}{b}\Big). \tag{13}$$

Here

$$C_{irjs}=-\frac{\lambda^2 ir(ir-js)}{4[(i+j)^2+\lambda^2(r+s)^2]^2}, \qquad A_{irjs}=\frac{\lambda^2 ir(ir+js)}{4[(i+j)^2+\lambda^2(r-s)^2]^2},$$

$$D_{irjs}=\frac{\lambda^2 ir(ir+js)}{4[(i-j)^2+\lambda^2(r+s)^2]^2}, \qquad B_{irjs}=-\frac{\lambda^2 ir(ir-js)}{4[(i-j)^2+\lambda^2(r-s)^2]^2}, \qquad \lambda=\frac{a}{b}.$$

Substituting (13) and (12) into the first equation (4) and implementing the procedure of the Bubnov–Galerkin method using the properties of the Dirac function [1], we obtain the following system of nonlinear integrodifferential equations for the unknowns $w_{kl}=w_{kl}(t)$:

$$\frac{\rho b^4}{Eh^2\pi^2}\ddot{w}_{kl}+\frac{4b^3}{\pi^2 ah^3E}\sum_{n=1}^{N}\sum_{m=1}^{M}\Big(\sum_{p=1}^{I}M_p\sin\frac{k\pi x_p}{a}\sin\frac{n\pi x_p}{a}\sin\frac{l\pi y_p}{b}\sin\frac{m\pi y_p}{b}\Big)\ddot{w}_{nm}$$

$$+\frac{\pi^2}{12(1-\mu^2)}\Big[\Big(\frac{k}{\lambda}\Big)^2+l^2\Big]^2(1-R^*)(w_{kl}-w_{0kl})$$

$$=\frac{16\alpha_{kl}hq}{kl\pi^4E}\Big(\frac{b}{h}\Big)^4-\frac{1}{h^2}\sum_{n,i,j=1}^{N}\sum_{m,r,s=1}^{M}a_{klnmirjs}w_{nm}(1-R^*)(w_{ir}w_{js}-w_{0ir}w_{0js}), \tag{14}$$

$$w_{kl}(0)=w_{0kl}, \qquad \dot{w}_{kl}(0)=\dot{w}_{0kl}, \qquad k=1,2,\ldots,N, \quad l=1,2,\ldots,M.$$

Here $\alpha_{kl}=1$ if k and l are odd and $\alpha_{kl}=0$ if k or l even; the coefficient $a_{klnmirjs}$ is determined from [14, 15].

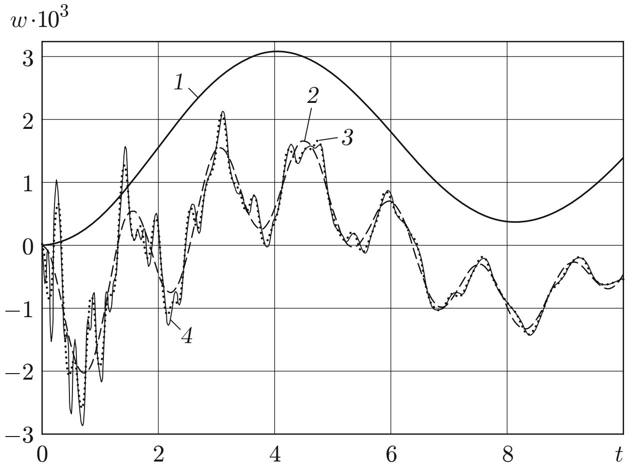

Fig. 2

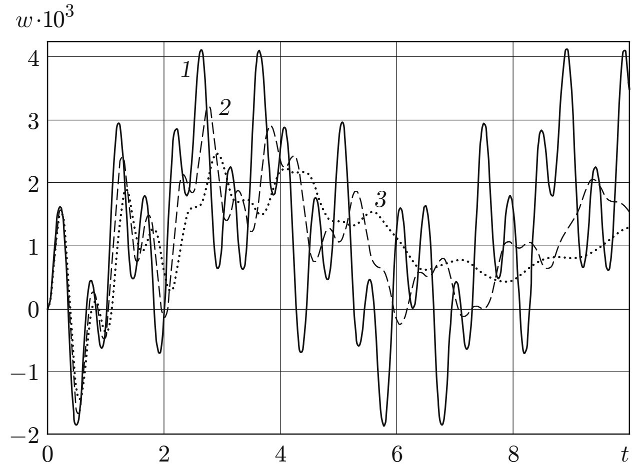

Fig. 3

Fig. 2. Deflection versus time for $N = 1$ (1), 3 (2), 7 (3), and 11 (4).

Fig. 3. Deflection versus time for $A = 0$ (1), 0.05 (2), and 0.1 (3).

Introducing the dimensionless quantity x_p/a, y_p/b, M_p/M_0, w_{kl}/h, w_{0kl}/h, $(q/E)(b/h)^4$, ωt, and $R(t)/\omega$ in (14) and retaining the former notation, we obtain

$$\sum_{n=1}^{N}\sum_{m=1}^{M} B_{klnm}\ddot{w}_{nm} + \frac{1}{4}\Big[\Big(\frac{k}{\lambda}\Big)^2 + l^2\Big]^2 (1 - R^*)(w_{kl} - w_{0kl})$$

$$= \frac{48\alpha_{kl}(1-\mu^2)}{kl\pi^6}\, q - \frac{3(1-\mu^2)}{\pi^2} \sum_{n,i,j=1}^{N}\ \sum_{m,r,s=1}^{M} a_{klnmirjs} w_{nm}(1 - R^*)(w_{ir}w_{js} - w_{0ir}w_{0js}), \qquad (15)$$

$$w_{kl}(0) = w_{0kl}, \qquad \dot{w}_{kl}(0) = \dot{w}_{0kl}, \qquad k = 1, 2, \ldots, N, \quad l = 1, 2, \ldots, M,$$

where $\omega = \sqrt{(4D/\rho h)(\pi/b)^4}$ is the fundamental vibration frequency, $M_0 = ab\rho h$ is the mass of the plate, and $B_{klnm} = 4\sum_{p=1}^{I} M_p \sin k\pi x_p \sin n\pi x_p \sin l\pi y_p \sin m\pi y_p$ for $k \neq n$, $l \neq m$; otherwise, $B_{klkl} = 1 + 4\sum_{p=1}^{I} M_p \sin^2 k\pi x_p \sin^2 l\pi y_p$.

System (15) was integrated using the numerical algorithm based on quadrature formulas which was described in Sec. 3. In the calculations, as the relaxation kernels we used the weakly singular Koltunov–Rzhanitsyn kernels (6). The calculations were performed in the Delphi algorithmic language.

Below, we give the results of calculations for various physical and geometrical parameters of the viscoelastic plate (Figs. 2–7). Unless otherwise indicated, the following initial parameter values were used: $w_0 = 10^{-3}$, $A = 0.05$, $\beta = 0.01$, $\alpha = 0.25$, $q = 0$, and $\lambda = 1$.

The convergence of the Bubnov–Galerkin method (Fig. 2) was studied. In the calculation of the deflection, the first seven harmonics were retained ($N = 7$ and $M = 1$). The calculations showed that a further increase in the number of terms did not have a significant effect on the vibration amplitude of the viscoelastic plate.

Figure 3 shows time dependences of the deflection at the center of an elastic plate (curve 1) and a viscoelastic plate (curves 2 and 3). It is evident that accounting for the viscoelastic properties of the plate material leads to the attenuation of the vibration process. In the initial period, the solution of the elastic and viscoelastic problems differ only slightly, but with time, the viscoelastic properties begin to have a significant effect. Studies have shown that

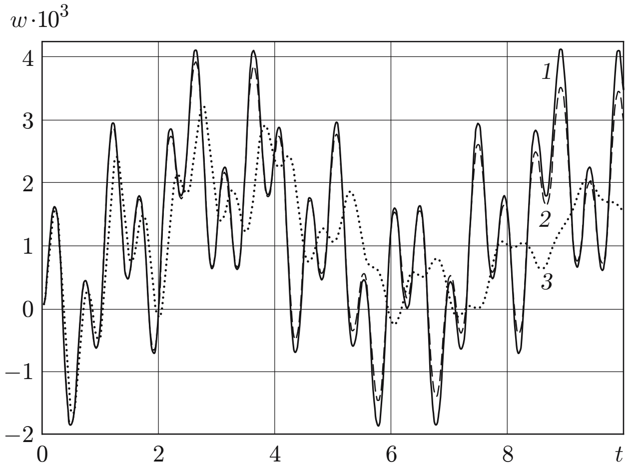

Fig. 4

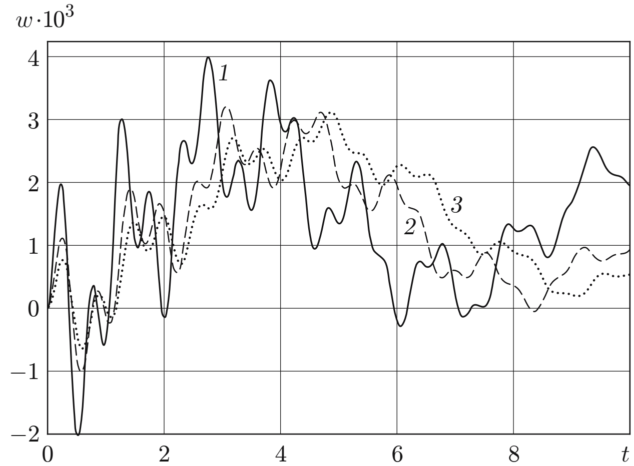

Fig. 5

Fig. 4. Deflection versus time for $M_1 = 0.1$: curve 1 refers to an elastic plate material and curves 2 and 3 refer to a viscoelastic material (curve 2 refers to the exponential relaxation kernel and curve 3 to the weakly singular Koltunov–Rzhanitsyn kernel).

Fig. 5. Deflection versus time for $M_1 = 0$ (1), 0.1 (2), and 0.2 (3).

an increase in the rheological parameter A and a decrease in the parameter α lead to a decrease in the vibration frequency and, hence, the amplitude.

The further calculations showed that variation in the third rheological viscosity parameter β ($0 < \beta < 1$) did not have a significant effect on the vibrations of the viscoelastic plate, which also confirms that the exponential relaxation kernels are unsuitable for calculations of the dynamic problems of viscoelastic systems.

Figure 4 shows the results of calculations of the strain of a square plate with a mass $M_1 = 0.1$ concentrated at the center in the absence of a transverse load ($q = 0$). It is evident that at the initial time, the results obtained using these kernels almost coincide; however, with time, their difference increases, and at $t = 10$, it is not larger than 30%. At the same time, the amplitudes obtained for the viscoelastic problem using the exponential relaxation kernel differ insignificantly from those obtained in the elastic formulation not only for the initial times but also over a fairly large time interval.

The effect of the mass concentrated at the center of the plate on the vibration process is shown in Fig. 5. It is evident that an increase in the concentrated mass leads to a decrease in the vibration amplitude and frequency. It should be noted that in the particular case where there is no concentrated mass at the center of the plate ($M_1 = 0$), the results coincide with the data given in [17].

The effect of the location of the concentrated mass on the vibration process (Fig. 6) was also studied. It was established that with distance of the concentrated mass from the center of the plate, the vibration frequency increased.

Figure 7 shows the results of calculations using various theories for a square plate with a mass $M_1 = 0.1$ concentrated at the center. In the absence of initial irregularities and external loads, the calculation results obtained using these theories coincide. However, in the presence of external loads and with increasing values of the initial irregularities, the dependences $w(t)$ obtained using the Kirchhoff–Love and Berger theories differ even at the initial times. Nevertheless, even under these conditions, the results obtained using the linear theory coincide for a fairly long time with those obtained using the Berger theory (the calculations for the Berger theory were performed for Eq. 5).

Conclusions. The above analysis of the results of studies of nonlinear dynamic problems of vibrations of viscoelastic plates with concentrated masses leads to the following conclusions.

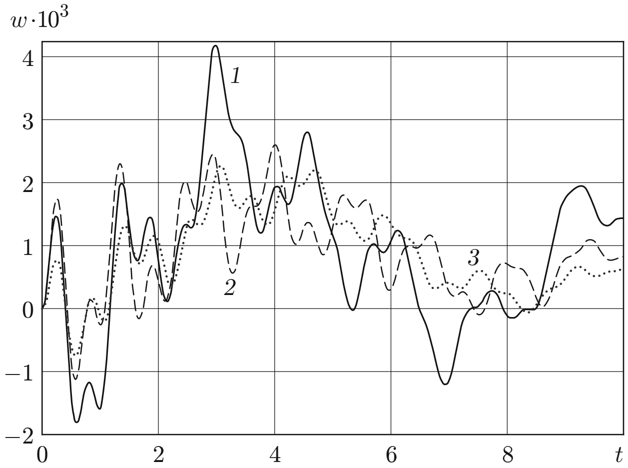

Fig. 6

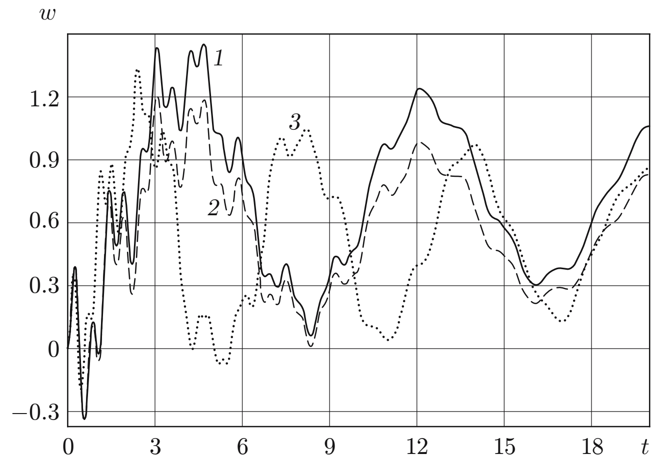

Fig. 7

Fig. 6. Deflection versus time for $M_1 = 0.1$, $y_1 = 1/2$, and $x_1 = 1/2$ (1), 1/3 (2), and 1/6 (3).

Fig. 7. Deflection versus time for $w_0 = 0.5$, $q = 1$, $M_1 = 0.1$, and $x_1 = y_1 = 1/2$: 1) linear theory; 2) Berger theory; 3) Kirchhoff–Love hypothesis.

The numerical results obtained using the exponential kernel as the relaxation kernel almost coincide with the results obtained in the elastic formulation. Therefore, as relaxation kernels one needs to use the Koltunov–Rzhanitsyn kernels, which contain a sufficient number of rheological parameters to obtain realistic numerical results for viscoelastic structures in good agreement with experimental data [10].

An increase in the concentrated mass leads to a more rapid decrease in the vibration amplitude and frequency.

In both the elastic and viscoelastic cases, the vibration frequency increases with distance of the concentrated mass from the center of the plate.

Depending on the values of the geometrical and physical parameters of plates, the corresponding theory (linear, Berger, or Kirchhoff–Love theory) should be used in the calculations. For vibrations of a square plate in the absence of external loads and initial irregularities, the results obtained using the indicated theories coincide. However, in the case of accounting for the initial irregularities of the plate and in the presence of external loads, a difference between the results obtained using these theories arises even at the initial times. For the present formulation of the problem, the classical Kirchhoff–Love theory is the most suitable.

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